"What a strange little object is the singularity with its strange properties and nonexistent definition. Yet the singularity promises to remain one of the most intriguing and disturbing aspects as gravitational theory for a long time to come." (Robert Geroch, 1967)

1. What is a spacetime singularity in GR?

2. The Schwarzscild solution

3. Singularity theorems in GR

4. Does the fact that singularities are a genetic feature of Einstein's field equations show that GR contains the seeds of its own destruction?

5. Does quantum gravity smooth away the singularities of classical GR?

Raychawdhuri's eq

expansion (θ), shear (σ), rotation (ω)

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}\xi^a\xi^b$$

(ξ ← normed 4-velocity)

ξ^a velocity field of a timelike congruence of geodesics

When the congruence is hypersurface $\perp$, $\omega_{ab} = 0$

EFE :

$$R_{ab}\xi^a\xi^b = 8\pi[T_{ab}\xi^a\xi^b + \tfrac{1}{2}T]$$

Strong Energy Condition (SEC) :

$$T_{ab}\xi^a\xi^b + \tfrac{1}{2}T \geq 0$$

For a perfect fluid, this means

$$e + 3p \geq 0, \quad e + p \geq 0$$

Upshot

For a non-rotating timelike geodesic congruence

$$\frac{d\theta}{d\tau} + \tfrac{1}{3}\Theta^2 \leq 0$$

Lemma

If initial $\Theta_0 < 0$ then $\Theta \to -\infty$ within a finite proper time $\tau \leq 3/|\Theta_0|$

Note: This does not mean that the geodesics cross in a finite time. Cf Misner spacetime.

Significance: If the geodesics are realized by dust matter, then there is an ∞ density singularity.

But in general the Lemma does not imply that a singularity develops – only that there is a caustic. This can happen even in singularity free Minkowski spacetime.

$\theta \to -\infty$ as q is approached implies that q is conjugate to p

i.e. there is an infinitesimally nearby geodesic γ' that intersects γ at p and q.

Let $N \subset M, g_{ab}$ be globally hyperbolic. And let γ be a timelike geodesic in N that maximizes proper time btw points p and q. Then γ contains

no point r conjugate to p btw p and q -- otherwise we could obtain a longer geodesic by "rounding off"

Lemma

For a globally hyperbolic M, g_{ab} if $p, q \in M$ can be joined by a timelike curve, then there is a timelike geodesic btw p and q that maximizes the length of timelike curves btw p and q

Putting all this together, we get a contradiction if the timelike geodesic through p for which $\theta_0 < 0$ can have a length $> 3/|\theta_0|$.

Upshot: There is a spacetime singularity in the sense of geodesic incompleteness!

General pattern of singularity thms:

<u>If</u>

1) Energy condition
2) Causality condition
3) initial/boundary condition

<u>then</u>

4) geodesic incompleteness

Does geodesic incompleteness imply

(i) curvature blow up?
Not necessarily.
E.g. Misner-Taub-NUT spacetime

(ii) missing points?
Not necessarily.
E.g. compact spacetime that is geodesically incomplete

Misner-Taub-NUT spacetime - incomplete timelike geodesics contained in a compact set.

Misner, "The Absolute Zero of Time," *Phys Rev.* **186** (1969), 3128-1333.

Three attitudes towards spacetime singularities:

(1) "Einstein avoids singularities."

(2) "Nature avoids Einsteinian singularitries"

Peter Bergmann: "It seems that Einstein was always of the opinion that singularities in classical field theory are intolerable. They are intolerable from the point of view of classical field theory because a singular region represents a breakdown of the postulated laws of nature. I think one can turn this argument around and say that a theory that involves singularities and involves them unavoidably, moreover, carries with it the seeds of its own destruction."

(3) "Nature and Einstein are subtle but tolerant."

Misner: " Singularities in solutions to EFE are a "source from which we can derive much valuable understanding of cosmology."

Kip Thorne: "From a purely philosophical standpoint it is difficult to believe that physical singularities are a fundamental and unavoidable feature of our universe. On the contrary, when faced with a theory which predicts the evolution of a singular state, one is inclined to discard or modify that theory rather than accept the suggestion that the singularity actually occurs in nature. Such was the case with Rutherford's theory of the atom …"

Brandenberger et al.: "The presence of singularities is an indication that GR is an incomplete theory … the [final] singularity implies that we cannot answer the question what will happen after the 'big crunch' or (in the case of an expanding universe) what was before the 'big bang'."

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